J. Fluid Mech. (1992), vol. 236, pp. 197–215
Printed in Great Britain

An analytic model of the generation of surface gravity waves by turbulent air flow

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(Received 30 November 1990 and in revised form 28 August 1991)

Turbulent air flow over a surface gravity wave of small amplitude is studied on the basis of a family of first-order closure models, of which the eddy viscosity model and Prandtl's mixing-length model are members. Results are obtained by the method of matched asymptotic expansions in three layers. The problem is modelled by taking into account the combined effects of turbulence and molecular viscosity, which accommodates a proper imposition of the boundary conditions at the wave surface. The detailed structure of the various wave-induced field variables throughout the flow is then investigated. In addition, it is found that the growth rate of the waves by wind depends on the turbulence model. In particular, the more sensitively the mixing length depends on the shear in the mean air flow, the higher the growth rate. The validity of the results we obtain is restricted to small drag coefficient and small phase speed. Comparisons are made with other theoretical studies and with recent laboratory and field observations.

1. Introduction

Generation of surface gravity waves by wind is an intriguing problem and has led to much controversy. This arose because, on the one hand, the first rational attempt towards a determination of the growth rate of waves by wind (Miles 1957) used a simple model for the turbulent air flow, where the direct effects of turbulence on the wave-induced motion in the air were disregarded. On the other hand, field measurements in the sixties and the beginning of the seventies (e.g. Dobson 1971) gave growth rates that were larger by an order of magnitude than Miles' prediction. The experimental determination of the growth rate is, however, far from straightforward as it involves the determination of a small phase difference between the pressure and the time derivative of the surface elevation.

Attempts to bridge the gap between observed and theoretical growth rates of waves by wind were directed towards the use of more realistic models of turbulent air flow over surface gravity waves on the one hand, and towards a more careful experimental determination of the above-mentioned phase difference on the other hand. Consequently, later field experiments by Snyder (1974) and by Snyder *et al.* (1981) show a better agreement between observed and theoretical growth rates, although, especially for the longer waves, the observed growth rates were still larger by a factor of two. The observed growth rate for high-frequency waves (Plant & Wright 1977) was found to be in agreement with Miles' theory (within a factor of 2).

A more realistic modelling of turbulent air flow over water waves started with the work of Chalikov (1976, 1978), Gent & Taylor (1976), Gent (1977) and Makin (1979). These authors introduced two new features. Firstly, the numerically determined growth rates were obtained from a fully nonlinear calculation. Secondly, the direct effect of turbulence on the wave-induced motion was taken into account. Air turbulence was modelled either by an eddy viscosity model or by a higher-order closure model (see also Al-Zanaidi & Hui 1984). Results of these more realistic models were, however, disappointing in that they still showed a substantial disagreement with observed growth rates for both high- and low-frequency waves. In fact, the agreement between Miles' theory and observations is better.

This rather curious situation calls for some explanation, which partially may be found in certain problems with both theory and experiment. The observed growth rate for high-frequency waves is obtained from the time series of the surface elevation (Plant & Wright 1977). Therefore, this growth rate not only represents the effect of wind on waves, but also effects of dissipation and nonlinear interactions. Janssen (1987) found that nonlinear interactions are at least as important as the effect of wind, giving rise to an uncertainty of a factor of two in the growth rate of high-frequency waves. However, the growth rate of low-frequency waves is uncertain also. As remarked by Makin (1988), the observed wave-induced pressure is extrapolated to the air–water interface assuming potential flow, which is not corroborated by the theoretical model calculations. Also, measurements of the height dependence of the wave-induced air pressure by Papadimitrakis, Hsu & Street (1986) show that for low-frequency waves, with phase speed of the order of the wind speed, the pressure does not always decay exponentially. According to Makin, Snyder *et al.* overestimate the growth rate of low-frequency waves by wind by a factor of two.

On the other hand, the models for turbulent air flow over surface gravity waves are not flawless either. Both the eddy-viscosity type of model and the higher-order closure models assume that the waves are only a slow perturbation to the turbulent air flow. Here, 'slow' means that the period of the surface wave is long compared to a typical relaxation time of the turbulence. As argued in §2 of this paper, the separation of timescales is not valid for low-frequency waves that have their critical layer far away from the water surface. For these waves the opposite case may be true, namely, that the period of the waves becomes shorter than the typical turbulent timescale. In that event, a completely different approach to the problem of the growth of waves by wind is needed, as for example in the paper Nikolayeva & Tsimring (1986) in which the effect of gustiness on wave growth is studied.

We conclude from this discussion that the problem of the growth of waves by wind is far from resolved. As Phillips mentioned in 1977, this problem provides a crucial test of our knowledge of turbulent air flow. For this reason we study in this paper the dependence of the results of the growth rate of gravity waves on the model of air turbulence. We restrict our attention to a family of first-order closure models, of which the eddy-viscosity model and Prandtl's mixing-length model are members.

In order to obtain a physically consistent model, which properly describes the behaviour of the various field variables throughout the flow, the effects of both turbulence and molecular viscosity are taken into account. Based on this model we study the linear stability of a plane-parallel shear flow in the air subject to wave-like perturbations at the air–water interface (see §2). The resulting eigenvalue problem is solved by the method of matched asymptotic expansions. For small drag coefficient, and small wave phase speed, one may distinguish three layers, namely, an inner layer, an intermediate layer and an outer layer. Molecular viscosity is dominant at the bottom of the inner layer and prevents the wind profile from becoming singular at the water surface. In §3 the three-layer structure of the problem is sketched in detail. In §4 the solutions in the various layers are matched and by application of the appropriate boundary conditions at the air–water interface the growth rate is determined.

Our results show that the growth rate of the waves indeed depends on the turbulence model. In particular, the more sensitively the mixing length depends on the shear in the air flow, the higher the growth rate.

It should be noted that Jacobs (1987) attempted to solve a similar problem for the eddy-viscosity model, where the effect of viscosity was disregarded. In that event, the problem has a two-layer structure, composed of the intermediate and outer layers mentioned above. Studying the solution in these layers only, this author was forced to assume that the stress in the surface layer depends on the small parameter (i.e. the drag coefficient) as well, a rather unusual circumstance since the stress is a control variable. It was a surprise to find that, although Jacob's method was not quite correct, his end result agrees with ours in case of the eddy-viscosity model.

In §5 uniformly valid expressions for the wave-induced field variables are derived. In §6 the results are compared with other studies (Stewart 1970; Gent & Taylor 1976; Snyder *et al.* 1981; Plant 1982; Hsu & Hsu 1983; Papadimitrakis *et al.* 1986; Makin 1989). Finally, in §7 we summarize our conclusions.

2. The governing equations

We consider the growth of surface gravity waves generated by a turbulent wind. The flow is assumed to be two-dimensional. The x-axis of a Cartesian coordinate system is parallel to the unidirectional, horizontal basic flow $\boldsymbol{U}$; the y-axis is vertical and increasing height corresponds to increasing y. The magnitude of the basic flow depends on height only. The air–water interface, when at rest, is located at $y = 0$. Both the air and the water are assumed to be incompressible, and the densities are constant.

The governing equations for air are of the form

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho_a}\boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot\{\nu(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^{\mathrm{T}})\}, \tag{1}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0, \tag{2}$$

where $\mathrm{D}/\mathrm{D}t$ is the material derivative, T denotes the transpose, $\boldsymbol{u} = (u, v)$ is the velocity, ρ_a is the density, p is the pressure, and ν is the kinematic viscosity.

The combined effects of molecular viscosity and turbulence are taken into account by assuming that the kinematic viscosity is given by

$$\nu = \nu_0 + \nu_e(y, t), \tag{3}$$

where ν_0 is the constant molecular viscosity and ν_e is the eddy viscosity which may be time-dependent.

To model the turbulence, the eddy viscosity is written in the general form

$$\nu_e = l^{1+n} u_*^{1-n} \left|\frac{\partial u}{\partial y}\right|^n, \tag{4}$$

where

$$l = \kappa(y - \xi_w), \tag{5}$$

with ξ_w the displacement of the air–water interface. In addition, the constant $n \geqslant 0$, κ is the von Kármán constant and u_* is the friction velocity. In the absence of waves ($\xi_w \to 0$) we have for $n = 0$ the eddy-viscosity model ($\nu_e = \kappa u_* y$), while for $n = 1$ we deal with Prandtl's mixing-length model (Tennekes & Lumley 1972). Equations (4) and (5) are a generalization to the case of a moving interface where we have made sure that the mixing length l vanishes at the interface.

The turbulence model (4), (5) is valid if a typical eddy turnover time (e.g. the time for the turbulence to respond to a change in forcing) is much smaller than the period of the waves. In that event, changes induced by the water surface in the air flow are slow and modelling of the momentum transport by means of (4) for a stationary turbulent flow seems justified. In other words, with ω the typical angular frequency of the wave, and T_t the timescale of the turbulent flow, we have the condition

$$\omega T_t \ll 1. \tag{6}$$

Here we take the eddy turnover time $T_t = L_t/u_*$, where L_t is chosen to be equal to the critical height y_c ($U(y_c = c$, where c is the phase speed of the wave) since in inviscid theory the critical layer plays a major role in the energy transfer from air to water. For a wind profile of the form

$$U(y) = \frac{u_*}{\kappa}\log\left(1+\frac{y}{y_0}\right), \tag{7}$$

with a Charnock relation for the roughness length y_0 according to

$$y_0 = \alpha\frac{u_*^2}{g}, \tag{8}$$

condition (6) becomes

$$\alpha\frac{u_*}{c}\{-1+\mathrm{e}^{\kappa c/u_*}\} \ll 1, \tag{9}$$

where we used the dispersion relation for deep-water gravity waves. Relation (9) boils down to the following restriction on the ratio of phase speed to friction velocity:

$$\frac{c}{u_*} \ll \frac{1}{\kappa}\log\left(\frac{25}{\alpha}\right),$$

where $\alpha \ll 1$. In other words, the turbulence model (4), (5) is only valid for high-frequency waves.

Although the theoretical calculations that follow are valid for arbitrary roughness length, we shall use the Charnock relation (8) when comparing our results with observations. In this connection, it should be realized, however, that the Charnock constant is not really a constant but depends (in the case of wind sea) on the wave age parameter c_p/u_*, where c_p is the phase speed corresponding to the peak frequency of the wind-sea spectrum (Donelan 1982; Janssen 1989; Maat, Kraan & Oost 1991). Here, we shall be mainly concerned with old wind sea and we take $\alpha = 0.0144$.

Using the set of equations (1)–(5) we study the stability of the basic flow, satisfying the relation

$$(\nu_0+\nu_e^{(0)})\frac{\mathrm{d}U}{\mathrm{d}y} = u_*^2, \tag{10}$$

where $\nu_e^{(0)}$ is the eddy viscosity in the absence of waves and where we impose $U(0) = 0$ because there is no mean current in the water. From (4) and (10) it follows that $\mathrm{d}U/\mathrm{d}y > 0$.

Following Jacobs (1987) we introduce dimensionless variables by defining a lengthscale $L = 1/k$, where k is the horizontal wavenumber, and a velocity scale V, where V is the wind speed at height L above the water surface. The coordinates are scaled by L, the velocities by V (where the dimensionless phase velocity is denoted by c), the time by L/V, the pressure by $\rho_a V^2$, and the viscosity by Lu_*. Furthermore, we define

$$\epsilon = u_*/V. \tag{11}$$

Since ϵ corresponds to the square root of the drag coefficient at height L, it will be regarded as a small parameter.

In terms of dimensionless quantities, the basic flow satisfies

$$(\nu_0 + \nu_e^{(0)})\frac{dU}{dy} = \epsilon, \quad \nu_e^{(0)} = \epsilon^{-n}(\kappa y)^{1+n}\left(\frac{dU}{dy}\right)^n. \tag{12}$$

To investigate the stability of the basic flow, we write $u = U + \tilde{u}$, $v = \tilde{v}$, $p = \tilde{p}$, where the tilde denotes a perturbation quantity. Furthermore, $\nu_e = \nu_e^{(0)} + \nu_e^{(1)}$, where $\nu_e^{(1)}$ denotes the perturbed eddy viscosity. Linearizing (1) and (2), and omitting the tildes, we obtain

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)u + \frac{dU}{dy}v = -\frac{\partial p}{\partial x} + \epsilon\left\{(\nu_0 + \nu_e^{(0)})\nabla^2 u + \frac{d\nu_e^{(0)}}{dy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\nu_e^{(1)}\frac{dU}{dy}\right)\right\}, \tag{13}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)v = -\frac{\partial p}{\partial y} + \epsilon\left\{(\nu_0 + \nu_e^{(0)})\nabla^2 v + 2\frac{d\nu_e^{(0)}}{dy}\frac{\partial v}{\partial y} + \frac{\partial}{\partial x}\left(\nu_e^{(1)}\frac{dU}{dy}\right)\right\}, \tag{14}$$

$$\frac{\partial u}{\partial x} + \frac{\partial x}{\partial y} = 0. \tag{15}$$

By introduction of the stream function Ψ according to

$$u = -\frac{\partial \Psi}{\partial y}, \quad v = \frac{\partial \Psi}{\partial x}, \tag{16}$$

the continuity equation is automatically satisfied, and by elimination of the pressure the equation for the stream function becomes

$$\epsilon(\nu_0 + \nu_e^{(0)})\nabla^4\Psi + 2\epsilon\frac{d\nu_e^{(0)}}{dy}\frac{\partial}{\partial y}\nabla^2\Psi + \frac{d^2U}{dy^2}\frac{\partial \Psi}{\partial x} - \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\nabla^2\Psi + \epsilon\frac{d^2\nu_e^{(0)}}{dy^2}\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\Psi$$
$$= \epsilon\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2}\right)\left(\nu_e^{(1)}\frac{dU}{dy}\right). \tag{17}$$

We consider a single Fourier component, with a dependence on x and t of the form $\exp\{i(x - ct)\}$. The water displacement, scaled by L, reads

$$\xi_w = f e^{i(x-ct)}, \tag{18}$$

where $|f|$ is the wave slope. As a consequence the perturbed eddy viscosity is given by

$$\nu_e^{(1)} = -\epsilon^{-n}\kappa^{1+n}\left\{ny^{1+n}\left(\frac{dU}{dy}\right)^{n-1}\frac{\partial^2\Psi}{\partial y^2} + f(1+n)\,y^n\left(\frac{dU}{dy}\right)^n\right\}. \tag{19}$$

The water is assumed to be incompressible, inviscid, and irrotational. There is no basic current and the water is infinitely deep. Its density ρ_w is a constant. The disturbances are assumed to vanish as $y \to -\infty$. At the interface of the air and water we require continuity of the horizontal velocity, the vertical velocity and the normal stress. Using the scaling introduced below (10), and defining

$$s = \rho_a/\rho_w, \tag{20}$$

the linearized continuity conditions (evaluated at $y = 0$) read, respectively,

$$-\frac{\partial \Psi}{\partial y} + \frac{dU}{dy} f = cf, \tag{21}$$

$$\Psi(0) = -cf, \tag{22}$$

$$2i\epsilon^2 f + p - 2\epsilon\nu \frac{\partial^2 \Psi}{\partial y\, \partial x} = \frac{f}{s}(c^2 - c_0^2), \tag{23}$$

where $c_0 = (gL/V^2)^{\frac{1}{2}}$ is the phase velocity of the water wave as $s \to 0$. The motivation for the choice of the no-slip condition will be described below (51).

Because of the importance of resonant interaction of the wave with the basic flow, we choose $c > 0$. In that case a critical layer develops around $U = c$. In view of (9) there is, on the other hand, an upper bound on the permitted values of the phase speed c in order that the turbulence model is valid.

3. The three-layer structure

We introduce the scale transformation

$$\zeta = y/\epsilon_1, \quad \epsilon_1 = \nu_0, \tag{24}$$

to describe the solution in the 'inner' layer $y = O(\epsilon_1)$.

In view of (12) and (24) the basic eddy viscosity and the basic velocity profile may be expressed as

$$\nu_e^{(0)} = \epsilon_1 \bar{\nu}_e(\zeta), \quad U = \frac{\epsilon}{\kappa} \bar{U}(\zeta), \tag{25}$$

where $\bar{\nu}_e$ and $\bar{U}$ depend on n and ζ only, and $d\bar{U}/d\zeta > 0$. The scaling introduced below (10) implies that $\bar{U}(1/\epsilon_1) = \kappa/\epsilon$. Consequently, $\epsilon_1 = o(1)$ as $\epsilon \to 0$. Thus, the limit $\epsilon \to 0$ corresponds to the limit of vanishing viscosity.

Making use of (12), (24) and (25), we deduce that

$$\frac{d\bar{U}}{d\zeta} \sim \frac{1}{\zeta}\left\{1 + \sum_{k=1}^{\infty} a_k \zeta^{-k}\right\} \quad \text{as} \quad \zeta \to \infty, \tag{26}$$

and

$$\bar{\nu}_e \sim \kappa\zeta\left\{1 + \sum_{k=1}^{\infty} a_k \zeta^{-k}\right\}^n \quad \text{as} \quad \zeta \to \infty, \tag{27}$$

where the constants a_k depend on n only. Integrating (26) with respect to ζ, and requiring that $\bar{U}(1/\epsilon_1) = \kappa/\epsilon$, we obtain

$$\epsilon_1 = e^{-(\kappa/\epsilon - \mu)}\{1 + O(e^{-\kappa/\epsilon})\} \quad \text{as} \quad \epsilon \to 0, \tag{28}$$

where $\mu = \mu(n)$ is the integration constant, determined from the requirement that $U(0) = 0$. Thus, the parameter ϵ_1 is transcendentally small with respect to ϵ, i.e. it vanishes exponentially as $\epsilon \to 0$. In what follows, transcendentally small terms will be denoted by TST.

Substituting (24) into (17), it is found that in the limit as $\epsilon \to 0$, with ζ fixed, the resulting equation reduces to the limit equation

$$\mathrm{L}_1 \Psi = 0, \tag{29}$$

where

$$\mathrm{L}_1 = \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left\{1+(1+n)\,\bar{\nu}_{\mathrm{e}}(\zeta)\right\}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}. \tag{30}$$

Making use of (27), it is easily shown that (29) has solutions of the form

$$\left.\begin{aligned} \Psi^{(1)}(\zeta) &= 1, \\ \Psi^{(2)}(\zeta) &= \zeta, \\ \Psi^{(3)}(\zeta) &= \zeta\log\zeta + O(\log\zeta) \quad \text{as} \quad \zeta\to\infty, \\ \Psi^{(4)}(\zeta) &\sim \zeta^2 \quad \text{as} \quad \zeta\to\infty. \end{aligned}\right\} \tag{31}$$

We now consider the 'outer' layer $y = O(1)$. In this layer, the basic eddy viscosity varies linearly with height, while the basic velocity varies logarithmically:

$$\nu_{\mathrm{e}}^{(0)} = \kappa y + \mathrm{TST}, \quad U = 1 + \frac{\epsilon}{\kappa}\log y + \mathrm{TST}. \tag{32}$$

At the bottom of the inner layer, on the other hand, where laminar viscosity is dominant, $\nu_{\mathrm{e}}^{(0)}$ is proportional to ζ^{1+n}, while U is linear.

In the limit as $\epsilon \to 0$, with y fixed, (17) reduces to the limit equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)\Psi = 0. \tag{33}$$

Applying the boundary condition $\Psi \to 0$ as $y \to \infty$, it is then found that the leading-order outer solution, which represents the inviscid mode, is proportional to $\exp(-y)$. In addition, (17) possesses two WKB solutions. The one vanishing as $y \to \infty$, and representing the 'viscous' mode, is of the form

$$\Psi_{\mathrm{WKB}} = y^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}\sigma(y/\epsilon)^{\frac{1}{2}}}\{1+o(1)\} \quad \text{as} \quad \epsilon\to 0, \tag{34}$$

where

$$\sigma = \left\{-\frac{4\mathrm{i}w}{\kappa(1+n)}\right\}^{\frac{1}{2}}, \quad w = 1-c. \tag{35}$$

It is now readily seen that none of the solutions (31) of the limit equation (29) for the inner layer matches with (34). Apparently, the matching can only be carried out via the solutions in an 'intermediate' layer between the inner and outer layers.

For the intermediate layer, the appropriate scale transformation reads

$$\eta = y/\epsilon. \tag{36}$$

Then we have

$$\nu_{\mathrm{e}}^{(0)} = \epsilon\kappa\eta + \mathrm{TST}, \quad U = 1 + \frac{\epsilon}{\kappa}\log\epsilon + \frac{\epsilon}{\kappa}\log\eta + \mathrm{TST}. \tag{37}$$

We now substitute (36) into (17), where use is made of (37). In the limit as $\epsilon \to 0$, with η fixed, the resulting equation then reduces to the limit equation

$$\mathrm{L}\Psi = 0, \tag{38}$$

where

$$\mathrm{L} = \frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\left\{\kappa(1+n)\,\eta\frac{\mathrm{d}^2}{\mathrm{d}\eta^2} - \mathrm{i}w\right\}. \tag{39}$$

Equation (38) has the linearly independent solutions

$$\left.\begin{aligned} \omega_1(\eta) &= 1, \\ \omega_2(\eta) &= \eta, \\ \omega_3(\eta) &= \mathrm{i}\sigma\eta^{\frac{1}{2}}K_1(\mathrm{i}\sigma\eta^{\frac{1}{2}}), \\ \omega_4(\eta) &= \mathrm{i}\sigma\eta^{\frac{1}{2}}I_1(\mathrm{i}\sigma\eta^{\frac{1}{2}}), \end{aligned}\right\} \tag{40}$$

where K_1 and I_1 are modified Bessel functions (Abramowitz & Stegun 1964). The solution $\omega_4(\eta)$ should be discarded because it tends exponentially to infinity as $\eta \rightarrow \infty$. The solution $\omega_3(\eta)$, on the other hand, vanishes exponentially as $\eta \rightarrow \infty$. As expected, it matches with the outer solution (34), because

$$\omega_3(\eta) \sim (\tfrac{1}{2}\mathrm{i}\pi\sigma)^{\frac{1}{2}}\,\eta^{\frac{1}{4}}\,\mathrm{e}^{-\mathrm{i}\sigma\eta^{\frac{1}{2}}} \quad \text{as} \quad \eta \rightarrow \infty. \tag{41}$$

The small-η expansion of $\omega_3(\eta)$ reads

$$\omega_3(\eta) = 1 + \mu_0\,\eta \log \eta + \mu_1\,\eta + O(\eta^2 \log \eta), \tag{42}$$

where $\mu_0 = -\frac{1}{4}\sigma^2$, $\mu_1 = -\mathrm{i}\sigma^2\{\frac{1}{4}\pi + \frac{1}{2}\mathrm{i}(\frac{1}{2} - \gamma + \log(2/\sigma))\}$.

Actually, there is a fourth layer, namely, the critical layer. It is situated around the point y_c where $U = c$. Integrating (26) with respect to ζ, and making use of (24), (25) and (28), it is found that

$$y_c \sim \mathrm{e}^{-\kappa w/\epsilon} \quad \text{as} \quad \epsilon \rightarrow 0. \tag{43}$$

The critical layer is situated at the top of the inner layer and at the bottom of the intermediate layer. Furthermore, the width is $O(\epsilon_1^{\mathrm{w}})$. It turns out, however, that the inner-layer solutions can be matched directly with the intermediate-layer solutions. Apparently, the critical layer is not important in this respect; cf. Townsend (1972).

4. Determination of the growth rate

4.1. *The inner layer*

In the inner layer, where the transformation (24) applies, the stream function is written as

$$\Psi_{\mathrm{inn}} = \Psi_\alpha(\epsilon;\zeta) + \frac{\epsilon_1}{\epsilon}\Psi_\beta(\epsilon;\zeta) + O\left(\frac{\epsilon_1^2}{\epsilon^2}\right). \tag{44}$$

Then the boundary conditions (21) and (22) imply that

$$\Psi_\alpha(\epsilon;0) = -cf, \quad \Psi'_\alpha(\epsilon;0) = \epsilon f, \tag{45}$$

$$\Psi_\beta(\epsilon;0) = 0, \quad \Psi'_\beta(\epsilon;0) = -\epsilon c f, \tag{46}$$

where the prime denotes differentiation with respect to ζ.

We now substitute (24) into (17). Furthermore, (44) is substituted into the resulting equation. This yields, making use of (25),

$$\mathrm{L}_1\,\Psi_\alpha = -\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\{\epsilon(1+n) f \zeta^n (\bar{U}')^{1+n}\}, \tag{47}$$

and

$$\mathrm{L}_1\,\Psi_\beta = \mathrm{i}\{(U-c)\,\Psi''_\alpha - U''\,\Psi_\alpha\}, \tag{48}$$

where L_1 is defined by (30).

From (12) and (25) we obtain

$$\{1+(1+n)\,\bar{\nu}_e(\zeta)\}\,\bar{U}'' = -\kappa(1+n)\,\zeta^n(\bar{U}')^{1+n}, \tag{49}$$

where

$$\bar{\nu}_e = \kappa\zeta^{1+n}(\bar{U}')^n. \tag{50}$$

Thus, (47) has a solution of the form

$$\Psi_\alpha = \left\{-c+\frac{\epsilon}{\kappa}\bar{U}(\zeta)\right\}f = (U-c)f, \tag{51}$$

which satisfies the boundary conditions (45).

Since the water is assumed to be inviscid, one of the continuity conditions at the interface between the air and water should be dropped. Since continuity of the vertical velocity and the normal stress are essential, there remains either the no-slip condition or the condition of continuity of the shear stress. It turns out that the no-slip condition is essential, which can be shown as follows.

The general solution of (47) is of the form

$$\Psi_\alpha(\epsilon;\zeta) = \frac{\epsilon}{\kappa}\bar{U}(\zeta)f + K_1\,\Psi^{(1)}(\zeta)+K_2\,\Psi^{(2)}(\zeta)+K_3\,\Psi^{(3)}(\zeta)+K_4\,\Psi^{(4)}(\zeta),$$

cf. (29), (31) and (49), where the constants K_i may depend on ϵ. We now write the asymptotic expansion of the above solution as $\zeta\to\infty$ in terms of the intermediate variable η, defined by (36), where we also make use of (24) and (28). Then it is found that, unless $K_2 = K_3 = K_4 = 0$, the resulting expansion (in terms of η) tends to infinity as $\epsilon_1\to 0$. Only when this limit is bounded, however, do we obtain physically meaningful results. Then the condition of continuity of the vertical velocity, which implies that $\Psi_\alpha(\epsilon;0) = -cf$, determines the remaining constant K_1 in Ψ_α. Since Ψ_α then reduces to (51), however, the no-slip condition (21) is automatically satisfied to leading order. In other words, the no-slip condition is the relevant third continuity condition.

In view of (51), equation (48) reduces to

$$\mathrm{L}_1\,\Psi_\beta = 0. \tag{52}$$

The solution of this equation is written as

$$\Psi_\beta = \epsilon\Psi_0(\zeta)+\epsilon^2\Psi_1(\zeta)+(\epsilon^3\log\epsilon)\,\Psi_2(\zeta)+\epsilon^3\Psi_3(\zeta)+\dots, \tag{53}$$

where the various $\Psi_i(\zeta)$ all satisfy the equation $\mathrm{L}_1\,\Psi_i = 0$. In view of (29) and (31), Ψ_i is of the general form

$$\Psi_i(\zeta) = A_i\,\Psi^{(1)}(\zeta)+B_i\,\Psi^{(2)}(\zeta)+C_i\,\Psi^{(3)}(\zeta)+D_i\,\Psi^{(4)}(\zeta). \tag{54}$$

Next we shall write the asymptotic behaviour of (44) as $\zeta\to\infty$ in terms of the variable η, defined by (36). Integrating (26) with respect to ζ yields the asymptotic expression for $\bar{U}$ as $\zeta\to\infty$. Substituting the relation $\zeta = \epsilon\eta/\epsilon_1$ into this expression, where ϵ_1 is given by (28), the solution (51) becomes

$$\Psi_\alpha \sim \left(w+\frac{\epsilon}{\kappa}\log\epsilon+\frac{\epsilon}{\kappa}\log\,\eta\right)f+\mathrm{TST}. \tag{55}$$

Now (55) is bounded in the limit as $\epsilon\to 0$, η fixed. Consequently, when the asymptotic behaviour of $(\epsilon_1/\epsilon)\,\Psi_\beta$ is written in terms of η, this limit should be bounded also. This leads to the condition that $D_i = 0$ in (54). Regarding the solution Ψ_0, the factor C_0 in (54) should vanish. Then the boundary conditions (46) imply that $A_0 = 0$, $B_0 = -cf$. The coefficients in Ψ_1, Ψ_2, Ψ_3, on the other hand, can only be determined by matching with the solution in the intermediate layer. We obtain:

$$\Psi_\alpha + \frac{\epsilon_1}{\epsilon}\Psi_\beta \sim wf + (\epsilon\log\epsilon)\frac{f}{\kappa} + \epsilon\left\{\frac{f}{\kappa}\log\eta + (\kappa C_1 - cf)\,\eta\right\}$$
$$+ (\epsilon^2\log\epsilon)\,(C_1 + \kappa C_2)\,\eta + \epsilon^2\{(B_1 - \mu C_1 + \kappa C_3)\,\eta + C_1\,\eta\log\eta\} + O(\epsilon^3\log^3\epsilon). \quad (56)$$

Substitution of (24) and (44) into (14) leads to the expression for the derivative of the pressure in the inner layer, namely,

$$p'_{\text{inn}} = \mathrm{i}\epsilon\{(1+\bar{\nu}_{\text{e}})\,\Psi'''_\alpha + 2\bar{\nu}'_{\text{e}}\,\Psi''_\alpha - n\bar{\nu}_{\text{e}}\,\Psi''_\alpha - \epsilon f(1+n)\,\zeta^n(\bar{U}')^{1+n}\} + \text{TST}. \quad (57)$$

With reference to (49), (50) and (51), expression (57) reduces to $p'_{\text{inn}} = \text{TST}$. Consequently, the pressure in the inner layer reads

$$p_{\text{inn}} = p_\alpha(\epsilon) + \text{TST}, \quad (58)$$

where p_α is independent of ζ.

4.2. *The outer layer*

In the outer layer, the stream function is written as

$$\Psi_{\text{out}} = \theta_0(y) + \epsilon\theta_1(y) + \ldots, \quad (59)$$

where the WKB solution (34) is omitted for the time being.

Substituting (59) into (17), and making use of (32), we obtain

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)\theta_0 = 0, \quad (60)$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}y^2} - 1\right)\theta_1 = -\frac{\theta_0}{\kappa w y^2} - \frac{\mathrm{i}n\kappa}{w}\left\{\frac{\mathrm{d}^2}{\mathrm{d}y^2} + 1\right\}\left(y\frac{\mathrm{d}^2\theta_0}{\mathrm{d}y^2}\right). \quad (61)$$

Requiring that $\Psi \to 0$ as $y \to \infty$, the solutions of (60) and (61) read

$$\theta_0 = F_0\,\mathrm{e}^{-y}, \quad (62)$$

$$\theta_1 = F_1\,\mathrm{e}^{-y} - \frac{F_0}{\kappa w}\mathrm{e}^{y}\,E_1(2y) + \frac{\mathrm{i}n\kappa F_0}{2w}(y^2 - y)\,\mathrm{e}^{-y}, \quad (63)$$

where F_0 and F_1 are constants to be determined, and E_1 denotes the exponential integral (Abramowitz & Stegun 1964).

The intermediate expansion of the outer solution (59) is obtained by writing the small-argument expansions of $\theta_i(y)$ in terms of η. Making use of (62), (63) and the relation $E_1(2y) = -(\gamma + \log 2) - \log y + 2y + O(y^2)$ as $y \to 0$, where γ is Euler's constant, it is found that

$$\Psi_{\text{out}} \sim F_0 + (\epsilon\log\epsilon)\frac{F_0}{\kappa w} + \epsilon\left\{-F_0\,\eta + F_1 + \frac{F_0}{\kappa w}(\gamma + \log 2 + \log\eta)\right\} + (\epsilon^2\log\epsilon)\frac{F_0}{\kappa w}\eta + O(\epsilon^2). \quad (64)$$

The constants in the intermediate expansion (56) of the inner solution, and those in the intermediate expansion (64) of the outer solution, are determined by matching these expansions with the solution in the intermediate layer. This will also determine the magnitude of the viscous mode, and the growth rate.

4.3. *The intermediate layer*

In the intermediate layer, where the scale transformation (36) applies, the stream function is written as

$$\Psi_{\text{int}} = \varphi_0(\eta) + (\epsilon \log \epsilon)\,\varphi_1(\eta) + \epsilon\varphi_2(\eta) + (\epsilon^2 \log \epsilon)\,\varphi_3(\eta) + \epsilon^2\varphi_4(\eta) + \dots. \tag{65}$$

Expressing (17) into the variable η, and making use of (37), the substitution (65) leads to a hierarchy of equations for the various φ_i.

The leading-order term φ_0 satisfies the equation

$$\mathrm{L}\varphi_0 = 0, \tag{66}$$

where L is defined by (39). The general solution of (66) is a linear combination of the basic solutions in (40), where it is recalled that the basic solution ω_4 should be discarded. The small-argument expansion of the general solution is obtained by making use of (42). Comparing this expansion with the intermediate expansion (56) of the inner solution yields

$$\varphi_0 = wf. \tag{67}$$

Now the equation for φ_1 reads

$$\mathrm{L}\varphi_1 = 0. \tag{68}$$

Proceeding in the same manner as before, matching of the general solution of (68) yields

$$\varphi_1 = f/\kappa. \tag{69}$$

Likewise, from the equation

$$\mathrm{L}\varphi_2 = \frac{\mathrm{i}wf}{\kappa\eta^2} - \frac{2(1+n)f}{\eta^3}, \tag{70}$$

we obtain

$$\varphi_2 = \frac{f}{\kappa}\log\eta - wf\eta, \tag{71}$$

while the coefficient C_1 in (56) becomes

$$C_1 = \left(\frac{2c-1}{\kappa}\right)f. \tag{72}$$

Matching the solution (65) with the intermediate expansion (64) of the outer solution, where use is made of (67), (69) and (71), yields

$$F_0 = wf, \quad F_1 = -\frac{f}{\kappa}(\gamma + \log 2), \tag{73}$$

which determines the outer solution (59) up to $O(\epsilon)$.

Substituting (36) and (37) into (13), we obtain the following expression for the pressure in the intermediate layer:

$$p_{\text{int}} = \frac{\mathrm{i}}{\epsilon}\left[\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{(1+n)\,\kappa\eta\frac{\mathrm{d}^2}{\mathrm{d}\eta^2}\right\} - \mathrm{i}w\frac{\mathrm{d}}{\mathrm{d}\eta}\right]\Psi + \left(\frac{\log\epsilon}{\kappa}\right)\frac{\mathrm{d}\Psi}{\mathrm{d}\eta} + \left(\frac{\log\eta}{\kappa}\right)\frac{\mathrm{d}\Psi}{\mathrm{d}\eta}$$
$$-\frac{\Psi}{\kappa\eta} - \frac{\mathrm{i}(1+n)f}{\eta^2} + \mathrm{i}\kappa\epsilon\left(\Psi - \eta\frac{\mathrm{d}\Psi}{\mathrm{d}\eta}\right) + \mathrm{TST}, \quad \Psi = \Psi_{\text{int}}. \tag{74}$$

In view of (65), (67), (69) and (71), this may be written in the alternative form

$$p_{\text{int}} = -w^2 f + (\epsilon \log \epsilon)\,p_1(\eta) + \epsilon p_2(\eta), \tag{75}$$

where p_1 and p_2 are determined by φ_1–φ_4. Note that (75) is bounded as $\epsilon \to 0$.

To calculate the growth rate, φ_3 and φ_4 should be known. The equations for these terms are of the form

$$\mathrm{L}\varphi_3 = 0, \tag{76}$$

$$\mathrm{L}\varphi_4 = -\mathrm{i}w^2 f - \frac{\mathrm{i}wf}{\kappa\eta}. \tag{77}$$

Matching the general solutions of (76) and (77) with (56) yields

$$\varphi_3 = \beta_3\,\eta, \quad \varphi_4 = \tfrac{1}{2}wf\eta^2 + \frac{f}{\kappa}\eta\log\eta + \beta_4\,\eta + 2\mathrm{i}(1+n)f\{-1+\omega_3(\eta)\}, \tag{78}$$

where β_3 and β_4 are still to be determined because not all parameters in (56) are known.

To determine (78) definitely, we proceed as follows. The asymptotic expression (as $\eta \to \infty$) for the term $\epsilon^2\varphi_4$ in (65) contains a term proportional to $\epsilon^2\eta^{\frac{1}{4}}\exp(-\mathrm{i}\sigma\eta^{\frac{1}{2}})$, as is deduced from (41) and (78). The outer expansion of this term matches with the viscous mode (in the outer layer), which is proportional to (34). Incorporating this mode into the outer solution (59), it is found that

$$\Psi_{\text{out}} = wf\,\mathrm{e}^{-y} + \epsilon f\left\{-\frac{1}{\kappa}(\gamma+\log 2)\,\mathrm{e}^{-y} - \frac{1}{\kappa}\mathrm{e}^{y}E_1(2y) + \tfrac{1}{2}\mathrm{i}n\kappa(y^2-y)\,\mathrm{e}^{-y}\right\}$$
$$+\epsilon^{\frac{7}{4}}(1+n)\,2\mathrm{i}f(\tfrac{1}{2}\mathrm{i}\pi\sigma)^{\frac{1}{2}}y^{\frac{1}{4}}\,\mathrm{e}^{-\mathrm{i}\sigma(y/\epsilon)^{\frac{1}{2}}} + \ldots, \tag{79}$$

cf. (62), (63) and (73).

The wave-induced horizontal velocity in the outer layer, denoted by u_{out}, is determined from (79) according to $u_{\text{out}} = -\mathrm{d}\Psi_{\text{out}}/\mathrm{d}y$, cf. (16). Then the intermediate expansion reads

$$u_{\text{out}} \sim -\frac{f}{\kappa\eta} + wf - (\epsilon\log\epsilon)\frac{f}{\kappa} - \epsilon f\left\{\frac{1}{\kappa}\log\eta + \frac{1}{\kappa}(-1+2\gamma+2\log 2)\right.$$
$$\left. -\tfrac{1}{2}\mathrm{i}n\kappa + w\eta + 2\mathrm{i}(1+n)\,(\tfrac{1}{2}\mathrm{i}\pi\sigma)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}\eta}(\eta^{\frac{1}{4}}\,\mathrm{e}^{-\mathrm{i}\sigma\eta^{\frac{1}{2}}})\right\} + \ldots. \tag{80}$$

Now we substitute (67), (69), (71) and (78) into the expression (65) for Ψ_{int}. Matching (80) with the wave-induced horizontal velocity in the intermediate layer, i.e. matching with $-\epsilon^{-1}\,\mathrm{d}\Psi_{\text{int}}/\mathrm{d}\eta$, finally determines β_3 and β_4 in (78). We obtain

$$\varphi_3 = \frac{f}{\kappa}\eta, \tag{81}$$

$$\varphi_4 = f\left[\tfrac{1}{2}w\eta^2 + \frac{1}{\kappa}\eta\log\eta + \left\{\frac{2}{\kappa}(-1+\gamma+\log 2) - \tfrac{1}{2}\mathrm{i}n\kappa\right\}\eta + 2\mathrm{i}(1+n)\,\{-1+\omega_3(\eta)\}\right]. \tag{82}$$

Then (75) becomes

$$p_{\text{int}} = -w^2 f + \epsilon wf\left\{\frac{2}{\kappa}(\gamma+\log 2) + 2\mathrm{i}\kappa\,(1+\tfrac{1}{4}n) + w\eta\right\} + \ldots. \tag{83}$$

Matching (83) with the pressure in the inner layer, given by (58), yields

$$p_{\text{inn}} = -w^2 f + \epsilon wf\left\{\frac{2}{\kappa}(\gamma+\log 2) + 2\mathrm{i}\kappa(1+\tfrac{1}{4}n)\right\} + \ldots. \tag{84}$$

4.4. *The growth rate*

The growth rate, ζ_r, is defined by

$$\zeta_r = 2\frac{\mathrm{Im}\,(c)}{c_0}, \tag{85}$$

where $c_0 = (gL/V^2)^{\frac{1}{2}}$ is the dimensionless phase velocity of the deep-water wave. It corresponds to the energy growth rate per radian (Miles 1957).

Substituting the expression (84) for the pressure at the water surface into the continuity condition (23) for the normal stress, we obtain the leading-order expression

$$\zeta_r = 2s\kappa\epsilon(1+\tfrac{1}{4}n)\left(\frac{1-c_0}{c_0^2}\right), \tag{86}$$

where s is defined by (20) and n is the free parameter in the turbulence model (4), (5).

For $n = 0$, corresponding to the eddy-viscosity model, the growth rate is the same as the one derived by Jacobs (1987). For $n = 1$, corresponding to Prandtl's mixing-length model, it becomes a factor of $\frac{5}{4}$ larger. The growth rate is larger as n increases. This shows that indeed the growth rate of the waves depends on the turbulence model.

To derive the expression (86) for the growth rate, it was assumed that the flow is two-dimensional. In the case of a three-dimensional flow, however, when the wind field makes an angle θ with the direction of wave propagation, the above analysis still applies, because the two-dimensional part of the wave-induced velocity field in the vertical planes parallel to the direction of wave propagation remains solenoidal. Retracing the analysis for the eddy-viscosity model, it is then found that

$$\zeta_r = 2s\kappa\epsilon\left(\frac{\cos\theta - c_0}{c_0^2}\right),$$

which shows that the growth rate is maximal for $\theta = 0$, corresponding to the case when the wind has the same direction as the wave. The above result also suggests that, when the phase speed is small enough, the wind transfers energy from the water to the air if $|\theta|$ exceeds some critical angle θ_c, with $\theta_c < \frac{1}{2}\pi$. In what follows, however, we only consider the two-dimensional model ($\theta = 0$).

5. The wave-induced field variables

We now derive uniformly valid expressions for the wave-induced velocities, u and v, and the wave-induced pressure p. The range of uniform validity is bounded from below by $y = y_1$, where $y_1 \gg \epsilon_1$. In view of (24) this means that the inner layer is excluded.

Let Σ denote either u, v or p. Then the uniformly valid expression for Σ, denoted by Σ_{un}, reads

$$\Sigma_{un} = \Sigma_{out} + \Sigma_{int} - \mathrm{cp}\,(\Sigma_{out}, \Sigma_{int}). \tag{87}$$

Here Σ_{out} denotes the outer solution, Σ_{int} denotes the intermediate solution and $\mathrm{cp}\,(\Sigma_{out}, \Sigma_{int})$ is the common part of Σ_{out} and Σ_{int}, consisting of the common terms that cancel out in the matching.

The outer solution for the wave-induced stream function is given by (79). When making a higher-order calculation, it turns out that the remainder term in (79) is

$O(\epsilon^2 \log \epsilon)$. The intermediate solution is determined from (65), (67), (69), (71), (81) and (82), where $\omega_3(\eta)$ is given by (40). The remainder term in (65) appears to be $O(\epsilon^3 \log^2 \epsilon)$. The velocities are found from (16). The outer solution for the pressure is determined from (13), (32) and (79). The intermediate solution is given by (83). From the above expressions we finally obtain

$$u_{\text{un}} = wf\,\mathrm{e}^{-y} - \epsilon f\left[\frac{1}{\kappa}(\gamma+\log 2)\,\mathrm{e}^{-y} - \frac{1}{\kappa}\mathrm{e}^{y} E_1(2y) + \frac{1}{\kappa}\frac{\mathrm{e}^{-y}}{y} - \tfrac{1}{2}in\kappa(1-3y+y^2)\,\mathrm{e}^{-y} + 2\mathrm{i}(1+n)\frac{\mathrm{d}\omega_3}{\mathrm{d}\eta}\right] + O(\epsilon^2 \log^2 \epsilon), \tag{88}$$

$$u_{\text{un}} = \mathrm{i}wf\,\mathrm{e}^{-y} + \mathrm{i}\epsilon f\left\{-\frac{1}{\kappa}(\gamma+\log 2)\,\mathrm{e}^{-y} - \frac{1}{\kappa}\mathrm{e}^{y} E_1(2y) + \tfrac{1}{2}in\kappa(y^2-y)\,\mathrm{e}^{-y}\right\} + O(\epsilon^{\frac{7}{4}}), \tag{89}$$

$$p_{\text{un}} = -w^2 f\,\mathrm{e}^{-y} + \epsilon wf\left[\left\{2\mathrm{i}\kappa + \tfrac{1}{2}in\kappa(1+y-y^2) + \frac{1}{\kappa}(\gamma+\log 2-\log y)\right\}\mathrm{e}^{-y} - \frac{1}{\kappa}\mathrm{e}^{y} E_1(2y)\right] + O(\epsilon^{\frac{5}{4}}). \tag{90}$$

To the order considered, it appears that the inner expansion of the outer solution for the pressure corresponds to the inner solution (84). In other words, the pressure at the water surface is obtained from the outer solution for the pressure by letting $y \to 0$. When this also applies to higher order, it can be shown that (86) is correct to $O(s\epsilon^2)$.

It should be noted that the range of uniform validity of the above expressions does not extend to infinity because of the secular terms. In fact, this range is limited to $y \leqslant y_2$, where $y_2 = O(\epsilon^{-\frac{1}{2}})$. For $n = 0$, however, the expressions for u and v are valid for all $y \geqslant y_1$. The expression for p, on the other hand, also contains a secular term for $n = 0$.

The secular terms can be removed by the method of multiple scales. We introduce the strained coordinate $y^+ = y(1 + a_1\epsilon + a_2\epsilon^2 + \dots)$ and the nonlinear scale $\xi = \epsilon y^2$. Writing the outer solution in the form

$$\Psi_{\text{out}} = \bar{\theta}_0(y^+, \xi) + \epsilon\bar{\theta}_1(y^+, \xi) + \dots \tag{91}$$

it is then found that

$$\Psi_{\text{out}} = wf \exp\left[-y + \frac{in\kappa\epsilon}{2w}(y^2 - y)\right] + O(\epsilon). \tag{92}$$

Apparently, the wave-induced velocities (and also the pressure) show an oscillatory dependence on height if $w = O(\epsilon)$.

6. Comparison with other studies

Reverting to the use of dimensional variables, the expression (86) for the growth rate becomes

$$\zeta_{\text{r}} = 2s\kappa\epsilon(1 + \tfrac{1}{4}n)\left(\frac{V-c}{V}\right)\left(\frac{V}{c}\right)^2, \tag{93}$$

where c denotes the dimensional phase speed of the water wave.

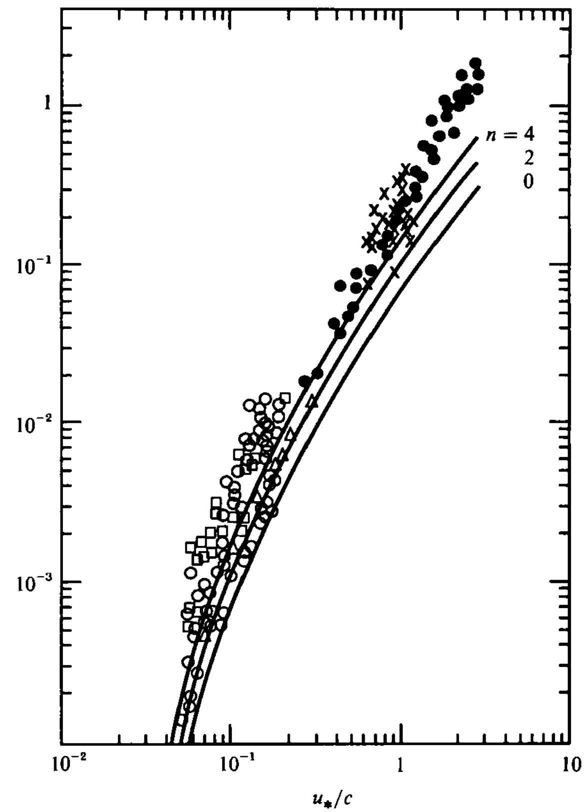

FIGURE 1. Growth rate (93) for various turbulence models ($n = 0, 2, 4$) against u_*/c, compared with measurements: △, Shemdin & Hsu (1967); ●, Larson & Wright (1975); ×, Wu, Hsu & Hsu (1977, 1979); ○, Snyder *et al.* (1981), fixed sensors; □, Snyder *et al.* (1981), wave-following sensor. Data compiled by Plant (1982).

At heights large compared with the roughness length y_0, the wind profile is approximately of the form $U = (u_*/\kappa)\log(y/y_0)$. The roughness length is related to the friction velocity according to the Charnock relation (8). Using the dispersion relation for deep-water waves, and requiring that $U(L) = V$ (see below (10)) we then obtain

$$\epsilon = -\frac{\kappa}{\log(\alpha u_*^2/c^2)}. \tag{94}$$

Furthermore, $c/V = (c/u_*)\epsilon$. Thus, the growth rate (93) is now fully determined by the parameter u_*/c.

The predicted growth rates are plotted in figure 1 for various values of n, where n denotes the free parameter in the turbulence model (4), (5). For $n = 0$ we deal with the eddy-viscosity model. In figure 1 we take $s = 0.00125$, $\kappa = 0.4$ and $\alpha = 0.0144$. The experimental data were compiled by Plant (1982). We conclude that, unless n is allowed to be large, the present theory underestimates the observed growth rates.

It should be noted, however, that the validity of the turbulence model (4), (5) is restricted to small phase speeds, see below (9). Moreover, the condition

$$\frac{V-c}{V} \gg \epsilon \tag{95}$$

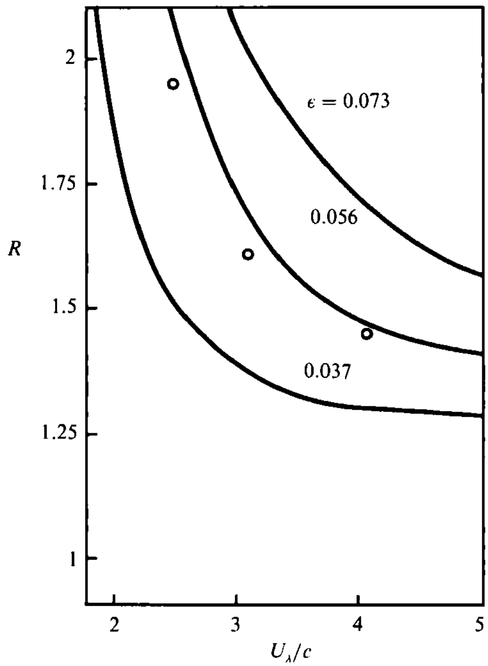

FIGURE 2. The open circles represent the ratio R of the growth rate of Gent & Taylor (1976) to the growth rate (93) for $n = 0$, $\epsilon = 0.05$. The wave slope is 0.157. The solid lines represent the corresponding ratio for the growth rate of Makin (1989) for $n = 0$ and various ϵ. The wave slope is 0.1.

must be satisfied in order that the analysis is valid. In other words, both ϵ and c/V must be small.

We took $\alpha = 0.0144$ (old wind sea) to obtain drag coefficients which are roughly in agreement with those reported by Snyder *et al.* (1981). We studied the sensitivity of our results by varying α by two orders of magnitude ($\alpha = 0.002$; $\alpha = 0.1$) and found that, in the range of validity of our theory, the growth rate only changed by a factor of three. The main reason for this insensitivity is that the growth rate only has a logarithmic dependence on the Charnock constant.

As mentioned before, the dependence of the predicted growth rate (93) on u_*/c is determined by (94). Unfortunately, the latter relation implies that ϵ is relatively large for small phase speeds, which violates condition (95). On the other hand, (94) presumably overestimates ϵ (i.e. the square root of the drag coefficient u_*^2/V^2) for small phase speeds. Therefore a smaller value of the Charnock constant α for the high-frequency waves would be more appropriate, leading to larger growth rates than those shown in figure 1.

Figure 2 shows the predicted growth rates compared with those obtained by Gent & Taylor (1976) and Makin (1989), based on nonlinear numerical calculations. In that figure, R (denoting the ratio of the nonlinear growth rate to the growth rate (93) for the eddy-viscosity model) is plotted against U_λ/c for various values of ϵ, where U_λ denotes the wind speed at a height of one wavelength. The agreement is better as the drag coefficient and the phase speed decrease compared to the wind speed. At increasing drag coefficient, or increasing wave speed, on the other hand, when condition (95) is violated, the agreement rapidly worsens.

Figure 3 shows a comparison of the predicted wave-induced vertical velocity

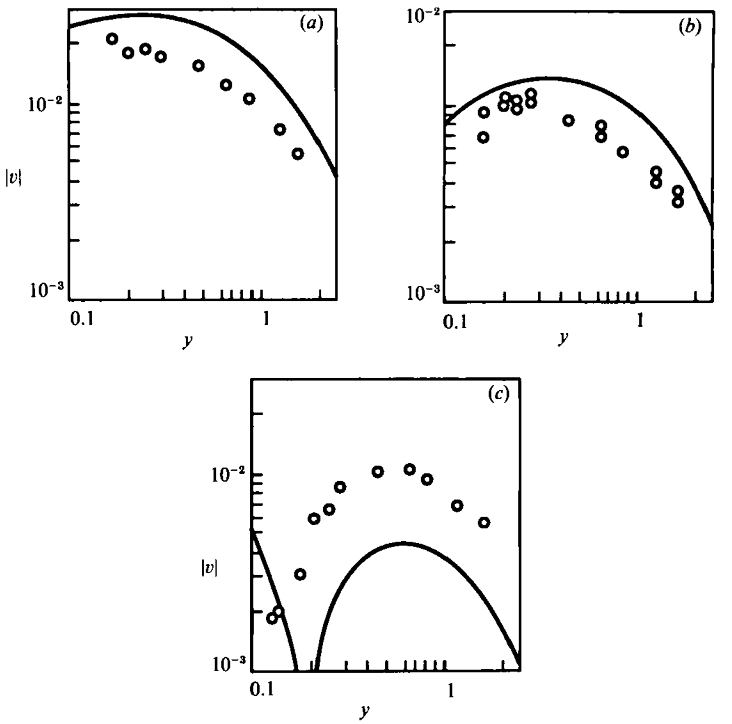

FIGURE 3. The amplitude of the predicted vertical velocity (89) with $n = 0$ (solid lines) compared with measurements (open circles) by Stewart (1970); (*a*) $c = 0.36, \epsilon = 0.052$; (*b*) $c = 0.52, \epsilon = 0.055$; (*c*) $c = 0.67$, $\epsilon = 0.056$.

profile (89) with laboratory measurements by Stewart (1970) at various (dimensionless) phase speeds and a small drag coefficient. All variables were non-dimensionalized according to the scaling below (10). The comparison was made for the eddy viscosity model. We conclude that the agreement is rather satisfactory at small phase speeds. At larger phase speeds, however, the agreement becomes poor, indicating once more that condition (95) is essential.

When the predicted velocities are compared with the laboratory measurements by Hsu & Hsu (1983), the same conclusions can be drawn. Even at relatively large dimensionless phase speeds ($c = 0.52$, for instance, corresponding to run 4 of their measurements) the amplitude of the predicted horizontal velocity (88) differs nowhere by more than 50 % from the measured amplitude.

The decay rate of the amplitude of the predicted air pressure (90) with height proves to be smaller than predicted by potential theory. For high wind speeds this is in agreement with the laboratory measurements by Papadimitrakis *et al.* (1986). In addition, the pressure phase shift from the wave trough is almost independent of height, which is again in agreement with measurements (Snyder *et al.* 1981; Papadimitrakis *et al.* 1986).

7. Summary of conclusions

The growth rate of water waves generated by wind that we obtained depends on the choice of the turbulence model. In particular, this rate is larger for Prandtl's mixing-length model than for the eddy-viscosity model. In case of the latter model the growth rate is the same as that derived by Jacobs (1987). This author studied a similar problem for the eddy-viscosity model but disregarded the effect of viscosity.

The present theory underestimates the measured growth rates. When compared with other theoretical studies, however, the agreement is much better, although nonlinear effects may be important. In this respect it should be noted that most theoretical studies underestimate the measured growth rates. To a certain extent this could be accounted for by the fact that the measured growth rates represent not only the energy transfer from the wind to the water wave. As shown by Gent & Taylor (1976), for example, a roughness length varying along the wave surface (simulating the presence of much smaller gravity and capillary waves riding on the dominant wave) significantly increases the growth rate compared to the case of a uniform roughness length, which may indicate the relevance of short wave–long wave interactions.

A comparison of the predicted vertical velocity profile with measurements shows a fairly reasonable agreement. However, the condition of small characteristic drag coefficient, and phase speeds which are small compared to the wind speed, is essential to the validity of the above results.

At small dimensionless phase speeds the decay rate of the predicted air pressure amplitude with height is smaller than predicted by potential theory. In addition, the pressure phase shift is almost independent of height. These results show again a qualitative agreement with laboratory measurements and field observations.

The turbulence models used in this paper are open to criticism. However, in preference to a refined (and inherently complex) model we introduced a family of relatively simple turbulence models which allow the derivation of closed-form expressions for the various variables and, consequently, could increase our insight into the flow structure. Then there remains the question which of these models is actually appropriate. In addition, it is of importance to investigate how far our results are modified by nonlinear effects. We expect that, especially for short gravity waves, when the critical layer is close to the water surface, nonlinearity may have an important effect on the growth rate and on the wave-induced flow.

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